A Ceteris Paribus Borda Solution to the Social Ranking Problem

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ABSTRACT

In our society, individuals are often rewarded based on their merits when they work in cooperation. Therefore, we need to design solutions that can fairly rank individuals based on their contribution to the success achieved by alternative groups or coalitions. In this paper, we focus on a novel social ranking solution where individuals are ranked based on the pairwise comparison of coalitions that differ for one single element (denoted in the related literature as *Ceteris Paribus* (CP-)comparison). We first introduce a set of axioms inspired by voting theory and social ranking to establish properties that a solution should satisfy when only a limited number of coalition is considered. Then, we show that our set of axioms uniquely characterizes a new solution that mimics a Borda rule computed over a coalitional preorder. These axioms include the one of *desirability*, a very well-established property in the setting of coalitional games but never used before in connection with a Borda rule. The other axioms, specifically *neutrality*, *separability*, and *cancellation*, are standard properties reflecting eponymous axioms in voting theory. Although a Borda score-based solution shows a cardinal feature, our axioms, which we also prove to be logically independent, are strongly rooted in an ordinal coalitional setting defined over a variable domain of possible coalitions. Connections with another solution from the literature on social ranking (CP-majority) reflecting a majority principle over the same type of CP-comparisons, are also illustrated.

KEYWORDS

Social Ranking, Coalitional Ranking, Borda, Axiomatic Approach, CP-Majority

1 INTRODUCTION

When trying to order individuals according to their performances, we might be compelled to consider comparisons between coalitions, other than direct one-on-one. This is because we believe that the importance of individuals should reflect the "sum" of their roles in the collective action provided by the different groups to which they belong, as in teams sports, where players may be crucial for some teams or squads, but not for others. And also because cooperation is particularly useful, as in politics or in scientific research, and a ranking of individuals that fairly mirrors their contribution to the success of various groups can be a useful tool for a efficient allocation of incentives aimed at rewarding most collaborative individuals. So, starting from groups' comparisons, we face the problem of grounding a ranking of individuals based on the information about the performance of their subsets, and taking into account their merits over multiple subsets. A formalized framework aimed at answering this question is known in the literature as the *social ranking problem* [8, 11, 15]. More precisely, a *solution* for this problem is a method to derive a ranking of individuals based on a transitive relation over a family of coalitions of individuals. Consider for instance the following instance of coalitional preorder on the set of individuals $\{1, \ldots, 8\}$ (rougly speaking, the symbol '$\succ$' represents 'stronger than' while the symbol '$\sim$' denotes 'equivalent strength'):

$$\{2, 3, 4\} \succ \{2, 6, 8\} \succ \{1, 3, 4\} \succ \{6, 3, 4\} \sim \{8, 3, 4\} \succ \{1, 6, 8\}.$$

Can we reasonably say whether 1 is better than 2 or than 6 by taking into account their multiple positions in the coalitional preorder (where the stronger, the better)?

Solutions to the social ranking problem can be linked to established approaches for addressing similar decision-making problems. Indeed, ordinal Banzhaf solution [13] comes from cooperative game theory [4], whereas CP-majority [11] is inspired by the classic voting rule of Condorcet [9]. We follow this dynamic here, and suggest an adaptation of the well-known Borda's voting rule. Following the approach of the CP-majority solution to the social ranking problem, we adopt the "*Ceteris Paribus*" point of view where a coalition S is regarded as a single voter that compares two individuals i and j through the comparison of $S \cup i$ and $S \cup j$. In the case of CP-majority solution, we then compare the number of coalitions that "prefer" an individual i over an individual j to establish an order over (i, j). The CP approach will allow us to refer to already known properties of Borda's voting rule and to construct a set of axioms to characterize this "CP-Borda" solution to the social ranking problem.

In this work we consider four axioms. Three of them, specifically, $Neutrality_{SR}$, $Separability_{SR}$ and $Cancellation_{SR}$, are reformulation in the context of social ranking of the eponymous axioms used in paper [21], while the fourth axiom is the well-known notion of *Desirability*, that has been widely studied in the literature of coalitional games [12, 14]. In words, Neutrality is a classical property in social choice theory stating that a social ranking should not depend on the names or labels of the individuals. Separability is another common property in voting theory, and for social ranking setting it is a natural way to consistently combine the information over disjoint families of coalitions. Cancellation is a property dealing with situations where the total number of pairwise set-comparisons in favor and against any individual is the same (precisely, CP-comparisons [11] where, as explained earlier, the relevant information for ranking two individuals consists of the pairwise comparisons of coalitions obtained by replacing one individual with the other *ceteris paribus*, that is, keeping the remaining members of the coalition unchanged). *Cancellation* is the only property involving a social ranking version of Borda score, and it subsumes the underlying equivalence between two coalitions that are indifferent and two others that are not comparable. Despite the fact that these properties mimic the corresponding axioms in [21], their adaptation to the social ranking problem is not straightforward. First, it demands the choice of a sensible criterion to identify a plausible notion of *voter* in terms of coalitions, as well as a consistent way to identify *candidates* among the individuals. Second, the variable domain of coalitions deeply affects the adaptation of the proof due to the manipulation of partial preference relations over coalitions.

As far as we know, this is the first time that the Desirability property, has been used in relation with a Borda-like rule. In our setting, this property specifies that a social ranking should align with the desirability relation: an individual i is more desirable than an individual j if each existing comparison between a set with i but not j and the same set obtained replacing j with i, results to be in favor of the set containing i. This property has been recently investigated in paper [1], showing that it can be considered as a building block of the most sensible social ranking solutions existing in the literature.

We start in the next Section 1.1 with a short discussion on the related literature. We continue introducing some preliminary notions on the social ranking problem in Section 2. Section 3 is devoted to the definition of a CP-Borda solution, and Section 3.1 to the axioms studied in this work. Some preliminary results related to the use of the axioms and the transformation of coalitional rankings are also introduced in Section 3.2. We present the axiomatic characterisation of the CP-Borda solution in Section 4, together with the analysis of the logical independence of the axioms in Section 4.1. Some connections between CP-majority and CP-Borda are discussed in Section 5. Section 6 concludes.

1.1 Related works

Prior to our work, several authors have proposed different social ranking rules and provided their axiomatic characterizations (see, for instance, paper [11] for CP-majority rule, [13] for the ordinal Banzhaf rule, and [2] for Lexcel, which is based on a lexicographic approach).

Some of our axioms share common philosophical underpinnings with those already used in the related literature. These connections will be discussed in Section 3.1. As already mentioned, a building block of CP-Borda is the notion of CP-comparison for coalitions, that has been originally introduced in paper [11](see articles [1, 2] for alternative axiomatic characterizations of the CP-majority). Let us also recall that there are articles analyzing social ranking rules from other perspectives, such as their manipulability [3, 20], core-stability [5], the possibility of coexistence of different axioms [6, 15, 19] and their applications [7, 17, 18].

Regarding Borda's voting rule this study is strongly related to article [21] as a primary axiomatic characterization but is also referencing [10] and [16] that provide slightly different ways to prove the axiomatic characterization of Borda's rule. These ways have been easier to adapt to our framework than the original proof in [21].

2 PRELIMINARIES

A **binary relation** R on a set N is a subset of $N \times N$. A binary relation on a set N that is both reflexive and transitive is a **preorder** on N.

In the following we will consider a set $N = \{1, ..., n\}$[1], whose elements can be referred to as **individuals** and its powerset 2^N, whose elements can be referred to as **coalitions**. A preorder on 2^N is called a **coalitional preorder** and will often be noted $\succcurlyeq$, with $\sim$ being its symmetric part and $\succ$ its asymmetric part. Given a finite set X, the set of all binary relations (resp. all preorders, resp. all complete preorders or rankings) on X is denoted $\mathcal{B}(X)$ (resp. $\mathcal{T}(X)$, resp. $\mathcal{R}(X)$).

The version of the **social ranking problem** that we consider here consists in the attribution of a total preorder (ranking) on any (finite) set N on the basis of a preorder on 2^N.

We denote $\mathcal{T}$ the set of all preorders on finite subsets of $\mathbb{N}$.

Definition 2.1. Given any finite subset $N \subseteq \mathbb{N}$, a **solution to the social ranking problem on N** is an application R^N from $\mathcal{T}$ to $\mathcal{R}(N)$. A **solution to the social ranking problem** is a set R of solutions to the social ranking on every set of individuals $N = \{1, ..., n\}$, $R = \{R^N, N = \{1, ..., n\}, n \in \mathbb{N}\}$. If R^N is a solution to the social ranking problem on N, for every preorder $\succcurlyeq$ in $\mathcal{T}$ we denote by $P^N(\succcurlyeq)$ the asymmetric part of $R^N(\succcurlyeq)$ and $I^N(\succcurlyeq)$ its symmetric part.

The Borda rule was characterized by Young ([21]) in the context of social choice theory (to answer the problem of how to elect the "best" candidates when complete preferences of voters over candidates are given). Young concludes his article by noting that the proofs can be generalized to the case of incomplete preferences. Our analysis of the CP-Borda solution for social ranking is inspired by this article and specifically addresses the issue of incomplete preference setting.

3 A BORDA SOLUTION AND SOME AXIOMS

Our version of a *Borda score* for the social ranking problem is based on *Ceteris paribus* (all other things being equal) comparisons, previously introduced in [11] for the definition of the CP-majority solution to the social ranking problem. According to this idea, given a set of individuals N, comparisons between two individuals i and j in N are made via comparisons of coalitions of the form $S \cup \{i\}$ and $S \cup \{j\}$, where $S \subseteq N$ is a specific coalition that does include neither i nor j. So, each specified coalition S is considered as a voter. We then derive *voters preferences* from the original coalitional preorder, as detailed later. Notice that a "voter-coalition" S cannot vote for individuals that are in S and therefore we obtain only preorders (not necessarily total). For this reason we are interested in applying Borda's rule on preorders, as suggested in [21].

In the following, let N be a set of individuals and $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$. We will denote by supp($\succcurlyeq$) the **support** of $\succcurlyeq$, defined

[1] We always consider N of this form, but since we only refer to solutions that don't take the label of individuals into account, we can use any finite set of individuals.

as the set of coalitions considered by this coalitional preorder,

$$\text{supp}(\succcurlyeq) = \{S \in 2^N, \exists T \in 2^N, T \succcurlyeq S \text{ or } S \succcurlyeq T\}.$$

Every element being in at least one coalition from the support of $\succcurlyeq$ is a **participant**,

$$\text{part}(\succcurlyeq) = \{i \in N | \exists S \in \text{supp}(\succcurlyeq), i \in S\}.$$

To be able to make a connection with Borda's social preference function [21] we need to define *voters* and *candidates*. The set of **voters** for $\succcurlyeq$ is defined as

$$\begin{aligned}\text{vot}(\succcurlyeq) \quad = \quad & \{S \in 2^N, \exists i, j \in N, i \neq j \\ & \text{and } S \cap \{i, j\} = \emptyset,\ S \cup \{i\}, S \cup \{j\} \in \text{supp}(\succcurlyeq) \\ & \text{and } S \cup \{i\} \succcurlyeq S \cup \{j\} \text{ or } S \cup \{j\} \succcurlyeq S \cup \{i\}\}.\end{aligned}$$

It follows that the set of **candidates** for $\succcurlyeq$ is defined as

$$\text{can}(\succcurlyeq) = \{i, \exists S \in \text{vot}(\succcurlyeq), i \notin S \text{ and } S \cup \{i\} \in \text{supp}(\succcurlyeq)\}.$$

To each voter S we associate a preorder $\geq^S$ such that, for every i, j in can($\succcurlyeq$), $S \cup \{i\} \succcurlyeq S \cup \{j\} \Rightarrow i \geq^S j$. $\geq^S$ is the preference preorder of the voter S, the multiset $\mathcal{P}_{\succcurlyeq} = \{\geq^S, S \in \text{vot}(\succcurlyeq)\}$ is the preference profile of $\succcurlyeq$.

Similarly as in the social preference frame [21], we define $\pi_{ij}(\succcurlyeq)$ for every participant i, j as $\pi_{ij}(\succcurlyeq) = |\{S \in \text{vot}(\succcurlyeq), i >^S j\}|$. We notice that if i or j isn't a candidate for $\succcurlyeq$, then $\pi_{ij}(\succcurlyeq) = 0$.

Example 3.1. Consider the set of individuals $N = \{1, 2, 3, 4, 5\}$ and the coalitional preorder $\succcurlyeq$ such that

$$\{2, 3, 4\} \succ \{1, 4\}, \{1, 2\} \succ \{1, 3\} \succ \{2, 3\} \sim \{1, 2, 3\} \succ \{1, 2, 4\},$$

$$\{4\} \succ \{3\} \succ \{1, 2, 5\}, \{1\} \succ \{2\}$$

then vot($\succcurlyeq$) = $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$, can($\succcurlyeq$) = $\{1, 2, 3, 4\}$ and voters preferences are $\{4\} \succ_\emptyset \{3\}$, $\{1\} \succ_\emptyset \{2\}$, $\{2\} \succ_{\{1\}} \{3\}$, $\{1\} \succ_{\{2\}} \{3\}$, $\{1\} \succ_{\{3\}} \{2\}$and $\{3\} \succ_{\{1,2\}} \{4\}$. We have $\pi_{12}(\succcurlyeq) = 2$, $\pi_{13}(\succcurlyeq) = \pi_{23}(\succcurlyeq) = \pi_{34}(\succcurlyeq) = \pi_{43}(\succcurlyeq) = 1$ and every other $\pi_{ij}(\succcurlyeq) = 0$.

Definition 3.2 (A Borda score). Given a coalitional preorder $\succcurlyeq$ in $\mathcal{T}(2^N)$ and an individual i in N, we define its **Borda score** in relation to $\succcurlyeq$, $\beta^{\succcurlyeq}(i)$, in the following way,

$$\beta^{\succcurlyeq}(i) = \sum_{j \in N} \pi_{ij}(\succcurlyeq) - \pi_{ji}(\succcurlyeq).$$

We notice that, by definition, the sum of all Borda scores in N is always equal to 0.

Definition 3.3 (CP-Borda). Given a set of individuals $N = \{1, ..., n\}$, the **CP-Borda solution to the social ranking problem on** N, R_B^N is the social ranking solution on N such that for every coalitional preorder $\succcurlyeq$ in $\mathcal{T}(2^N)$ and every individuals i and j in N, $iR_B^N(\succcurlyeq)j$ if $\beta^{\succcurlyeq}(i) \geq \beta^{\succcurlyeq}(j)$. As before, we call $P_B^N(\succcurlyeq)$ the asymmetric part, and $I_B^N(\succcurlyeq)$ the symmetric part of $R_B^N(\succcurlyeq)$.

$R_B = \{R_B^N, N = \{1, ..., n\}, n \in \mathbb{N}\}$ is the **CP-Borda solution to the social ranking problem**.

Since Borda score is well defined for every individual and takes values in $\mathbb{Z}$, CP-Borda social ranking solution produces a total preorder on the set of individuals and is indeed a solution for the social ranking problem. If N is obvious, we will abusively use R_B for R_B^N.

Example 3.4. In the Example 1, $\beta(1) = 2$, $\beta(2) = 0$, $\beta(3) = -2$, $\beta(4) = 0$ and $\beta(5) = 0$, so

$$1P_B(\succcurlyeq)2I_B(\succcurlyeq)4I_B(\succcurlyeq)5P_B(\succcurlyeq)3.$$

3.1 Axioms

We now introduce and discuss four axioms for a social ranking solution that would mimic those used to characterize Borda's social preference function in [21]. We start with a well-established notion of Neutrality for social ranking solutions. We first need some further notation. A bijection $\sigma : N \to N$ is said a permutation σ on N. For every element $\succcurlyeq$ of $\mathcal{T}(2^N)$ we denote as $\sigma(\succcurlyeq) \in \mathcal{T}(2^N)$ a coalitional preorder such that $S\sigma(\succcurlyeq)T \Leftrightarrow \sigma(S) \succcurlyeq \sigma(T)$ such that $S\sigma(\succcurlyeq)T \Leftrightarrow \sigma^{-1}(S) \succcurlyeq \sigma^{-1}(T)$ for all $S, T \in \text{supp}(\succcurlyeq)$ and where $\sigma(S)$ and $\sigma(T)$ denote the image of S and T through σ, respectively. With a slight abuse of notation, for any solution to the social ranking problem R, $\sigma(R^N(\succcurlyeq))$ is a total preorder on N such that $i\sigma(R^N(\succcurlyeq))j \Leftrightarrow \sigma^{-1}(i)R^N(\succcurlyeq)\sigma^{-1}(j)$.

It is common in decision procedures to want to ensure a notion of neutrality towards alternatives or individual's labels.

Definition 3.5 (Neutrality$_{SR}$). A social ranking solution R satisfies **Neutrality**$_{SR}$ if for every set of individuals N, every element $\succcurlyeq$ of $\mathcal{T}(2^N)$ and every permutation σ on N it holds that

$$\sigma(R^N(\succcurlyeq)) = R^N(\sigma(\succcurlyeq)).$$

Example 3.6. Consider $\succcurlyeq$ as in Example 3.1 and a permutation σ on N such that $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 4$, $\sigma(4) = 2$ and $\sigma(5) = 5$. We have that $\sigma(\succcurlyeq)$ is such that $\{1, 2, 4\} \succ \{2, 3\}, \{1, 3\} \succ \{3, 4\} \succ \{1, 4\} \sim \{1, 3, 4\} \succ \{1, 2, 3\}, \{2\} \succ \{4\} \succ \{1, 3, 5\}, \{3\} \succ \{1\}$. Let R^N be a social ranking solution that satisfies Neutrality$_{SR}$ and that on $\succcurlyeq$ yields the ranking $1P^N(\succcurlyeq)2P^N(\succcurlyeq)3I^N(\succcurlyeq)4P^N(\succcurlyeq)5$.
By Neutrality$_{SR}$ of R^N on $\sigma(\succcurlyeq)$, $P^N(\sigma(\succcurlyeq))$ must be such that $3P^N(\sigma(\succcurlyeq))1\ P^N(\sigma(\succcurlyeq))4\ I^N(\sigma(\succcurlyeq))2P^N(\sigma(\succcurlyeq))5$.

Neutrality$_{SR}$ has already been used in axiomatic characterizations of solutions to the social ranking, for instance in [8]. Other formulations exist such as the Neutrality axiom introduced in [11], but CP-Borda does not satisfy this version.

We say that $\succcurlyeq$ is **coherent** with $\succcurlyeq'$ and write "$\succcurlyeq \supseteq \succcurlyeq'$" if for all i and j in X it holds that $i \succcurlyeq' j \Rightarrow i \succcurlyeq j$, and $i \succ' j \Rightarrow i \succ j$. Given two elements of $\mathcal{T}(X)$, $\succcurlyeq$ and $\succcurlyeq'$, we denote by $\succcurlyeq \cap \succcurlyeq'$ the element of $\mathcal{T}(X)$ such that for every i and j in X, $i(\succcurlyeq \cap \succcurlyeq')j$ if $i \succcurlyeq j$ and $i \succcurlyeq' j$.

Given a set of candidates N and two elements $\succcurlyeq_1$ and $\succcurlyeq_2$ in $\mathcal{T}(2^N)$, if supp($\succcurlyeq_1$) $\cap$ supp($\succcurlyeq_2$) $= \emptyset$ we can define the set

$$\succcurlyeq_1 \Cup \succcurlyeq_2 = \{\succcurlyeq \in \mathcal{T}(2^{N'}), \text{supp}(\succcurlyeq) = \text{supp}(\succcurlyeq_1) \cup \text{supp}(\succcurlyeq_2), N' \supseteq N$$
$$\text{and } \succcurlyeq \supseteq \succcurlyeq_1, \succcurlyeq \supseteq \succcurlyeq_2\}.$$

Definition 3.7. We will denote by $\succcurlyeq_1 \boxplus \succcurlyeq_2$ the element of $\succcurlyeq_1 \Cup \succcurlyeq_2$ that is obtained without adding any further relations to those already defined by $\succcurlyeq_1$ and $\succcurlyeq_2$. It is minimal relatively to $\supseteq$ in the sense that if $\succcurlyeq$ is in $\succcurlyeq_1 \Cup \succcurlyeq_2$ then $\succcurlyeq \supseteq \succcurlyeq_1 \boxplus \succcurlyeq_2$.

Definition 3.8 (Separability$_{SR}$). A social ranking solution R satisfies **Separability**$_{SR}$ if for every set of individuals N and every elements $\succcurlyeq_1$, $\succcurlyeq_2$ of $\mathcal{T}(2^N)$ such that there is no coalition S and individuals i, j such that $S \cup \{i\}$ is in supp($\succcurlyeq_1$) and $S \cup \{j\}$ is in

$\text{supp}(\succcurlyeq_2)$, then for every element $\succcurlyeq$ of $\succcurlyeq_1 \Cup \succcurlyeq_2$, if $\succcurlyeq$ is in $\mathcal{T}(2^{N'})$ it holds that

$$R^{N'}(\succcurlyeq) \;\supseteq\; R^N(\succcurlyeq_1) \cap R^N(\succcurlyeq_2).$$

Example 3.9. Consider the set of individuals $N = \{1,2,3,4\}$ and the coalitional preorder $\succcurlyeq$ such that: $\{2,3,4\} \succ \{1,4\} \succ \{1,2\} \succ \{1,3\} \succ \{2,3\} \sim \{1,2,3\} \succ \{1,2,4\}$, $\{4\} \succ \{3\}$, and two other coalitional preorders $\succcurlyeq_1$ and $\succcurlyeq_2$ such that: $\{2,3,4\} \succ_1 \{1,4\}, \{1,2\} \succ_1 \{1,3\} \succ_1 \{2,3\} \sim_1 \{1,2,3\} \succ_1 \{1,2,4\}$, and $\{4\} \succ_2 \{3\}$. So, $\succcurlyeq \in \succcurlyeq_1 \Cup \succcurlyeq_2$.
Let R^N be a social ranking solution that satisfies Separability$_{SR}$ and such that, for instance, $3I^N(\succcurlyeq_1)4$ and $3P^N(\succcurlyeq_2)4$. By Separability$_{SR}$ of R^N we must have $3P(\succcurlyeq)4$. Notice that there is no coalition S and individuals i, j such that $S \cup \{i\}$ is in $\text{supp}(\succcurlyeq_1)$ and $S \cup \{j\}$ is in $\text{supp}(\succcurlyeq_2)$. On the contrary, if we consider another ranking $\succcurlyeq_3$ such that $\{4\} \succ_3 \{3\} \succ_3 \{3,4\}$, Separability$_{SR}$ of R^N would not apply to elements of $\succcurlyeq_1 \Cup \succcurlyeq_3$, for taking $S = \{3\}$, $S \cup \{1\}$ is in $\text{supp}(\succcurlyeq_1)$ and $S \cup \{4\}$ is in $\text{supp}(\succcurlyeq_3)$.

Separability, sometimes called consistency or reinforcement, has been introduced by H.P. Young in [21]. It is now a common considered axiom in the study of voting rules. To combine two coalitional preorders, Separability$_{SR}$ axiom requires that the respective supports of $\succcurlyeq_1$ and $\succcurlyeq_2$ are disjoint, so that we are able to define the union of two preorders, but also that we're neither considering the same voter multiple times nor creating a new voter. Compared to the Consistency axiom introduced and studied by T. Suzuki and M. Horita in [19], these constraints on the structure of the two coalitional preorders $\succcurlyeq_1$ and $\succcurlyeq_2$ reduce the number of situations to which Separability$_{SR}$ applies. Under Consistency, coalitions need not to reflect the notion of "voter" we adopt in our framework using CP-comparisons. So, in this sense Consistency is stronger than Separability (and CP-Borda does not satisfy it). It is easy to produce instances where CP-Borda (as well as CP-majority) does not satisfy Consistency (for more details, see Proposition 2, p.560 in [19]). Alternative definitions of Borda scores for social ranking (notrooted on the notion of CP-comparison) have been also studied in the same article [19].

In Young's axiomatic characterization [21], Faithfulness is used to ensure the "good ordering" of the ranking obtained by the Borda method. We replace it by the Desirability$_{SR}$ axiom in our axiomatic study, to have more flexibility when handling incomplete preferences.

Given a set of individuals N and a coalitional preorder $\succcurlyeq$ in $\mathcal{T}(2^N)$, we say that an individual i of N is **(resp. strictly) more desirable** than an individual j in N if for every voter S in $\text{vot}(\succcurlyeq)$, $S \cap \{i,j\} = \emptyset$, $S \cup \{i\} \succcurlyeq S \cup \{j\}$ (resp. and there is at least one voter T in $\text{vot}(\succcurlyeq)$ such that $T \cup \{i\} \succ T \cup \{j\}$).

Definition 3.10 (Desirability$_{SR}$). A social ranking solution R satisfies **Desirability$_{SR}$** if for every set of individuals N, every element $\succcurlyeq$ of $\mathcal{T}(2^N)$ and every individuals i, j in N it holds that

$$i \text{ is (resp. strictly) more desirable than } j \Rightarrow iR^N(\succcurlyeq)j \text{ (resp. } iP(\succcurlyeq)j).$$

Example 3.11. Consider again the coalitional preorder $\succcurlyeq$ of Example 3.1. We have that, for instance, individual 1 is strictly more desirable than 2 for $\{1,3\} \succ \{2,3\}$ and $\{1\} \succ \{2\}$, and these are the only CP-comparisons for 1 and 2, via voters $\{3\}$ and $\emptyset$, respectively. So, a solution satisfying Desirability$_{SR}$ should rank 1 strictly better than 2. Instead, there are two voters that allow to compare individuals 3 and 4, precisely, coalitions $\emptyset$ and coalition $\{1,2\}$, that cast votes opposite to each other. It follows that no desirability relation holds between 3 and 4, and we have no element to establish how a solution satisfying Desirability$_{SR}$ orders 3 and 4.

Desirability is a widely studied notion for coalitional games [12, 14] and the Desirability axiom for social ranking solutions is studied in [1] to prioritize an individual that systematically performs better than another individual in all CP-comparisons. This axiom has been used in [1] for the axiomatic characterization of other solutions to the social ranking problem, like CP-majority and other lexicographic solutions.

In the following, for every finite set N, we denote $\sim_N$ the binary relation R such that xRy and yRx for every x, y in N.

Definition 3.12 (Cancellation$_{SR}$). A social ranking solution R satisfies **Cancellation$_{SR}$** if for every set of individuals N and every element $\succcurlyeq$ of $\mathcal{T}(2^N)$ such that $\pi_{ij}(\succcurlyeq) = \pi_{ji}(\succcurlyeq)$ for every i, j in N, then it holds that

$$R^N(\succcurlyeq) \;= \sim_N .$$

Example 3.13. Consider the set of individuals $N = \{1,2,3,4\}$ and the coalitional preorder $\succcurlyeq$ such that

$$\{2,3,4\} \sim \{1,2\} \sim \{2,3\} \sim \{1,2,3\} \succ \{1,2,4\}, \{4\} \succ \{3\}.$$

We have $\pi_{ij}(\succcurlyeq) = 0$ for every $i, j \in N$. A solution satisfying Cancellation$_{SR}$ should rank indifferent all the individuals in N.

As far as we know, the property of Cancellation$_{SR}$ has not been previously introduced or used to characterize any other social ranking solution. Its counterpart for voting procedures, Cancellation property, has also been introduced by Young in [21].

Let R be a social ranking solution that satisfies Separability$_{SR}$ and Cancellation$_{SR}$. Let N and N' be two sets of individuals with $N \subseteq N'$ and $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$. Since $N \subseteq N'$, $\succcurlyeq$ is also an element of $\mathcal{T}(2^{N'})$. We show that $R^{N'}$ is coherent with R^N, meaning that considering a larger set of individuals to rank does not impact the ranking on the smaller set of individuals. We denote $\emptyset$ the empty binary relation. Notice that by Cancellation$_{SR}$, $R^N(\emptyset) = \sim_N$. Then, since $\succcurlyeq$ is also an element of $\succcurlyeq \Cup \emptyset$, then by Separability$_{SR}$, it follows that

$$R^{N'}(\succcurlyeq) \supseteq R^N(\succcurlyeq) \cap R(\emptyset) = R^N(\succcurlyeq) \cap \sim_N = R^N(\succcurlyeq).$$

This fact further justifies a slight abuse of notation using R to denote a social ranking solution satisfying Separability$_{SR}$ and Cancellation$_{SR}$, without specification on the set of individuals.

It is straightforward to show that CP-Borda satisfies the axioms presented in this section.

LEMMA 3.14. *CP-Borda solution to the social ranking problem satisfies Neutrality$_{SR}$, Separability$_{SR}$, Desirability$_{SR}$ and Cancellation$_{SR}$.*

The main result of this paper is that this set of axioms is in fact sufficient to characterize CP-Borda solution to the social ranking problem. To be able to prove this result we first need to introduce new tools in the form of transformations on coalitional preorders.

3.2 Transformations on coalitional preorders

To be able to duplicate voters and use axioms Separability$_{SR}$ and Cancellation$_{SR}$ more easily we define some transformations on coalitional preorders.

Definition 3.15. Let N be a set of individuals, X a finite subset of $\mathbb{N}$ and $\succcurlyeq$ be a preorder of $\mathcal{T}(2^N)$. Let $N' = \{1, ..., n'\}$ be the smallest set of individuals that contains X. We define the binary relation $\succcurlyeq^X$ on $2^{N'}$ in the following way,

$$\forall S, T \in 2^N, S \succcurlyeq T \Rightarrow S \cup X \succcurlyeq^X T \cup X.$$

Relation $\succcurlyeq^X$ is defined on elements of $2^{N'}$, and is transitive because $\succcurlyeq$ is transitive, so, it is an element of $\mathcal{T}(2^{N'})$. This transformation will be used to duplicate a coalitional preorder using new voters.

Definition 3.16. Let N be a set of individuals and $\succcurlyeq$ be a preorder of $\mathcal{T}(2^N)$. We define the binary relation $-\succcurlyeq$ (called the **inverse** of $\succcurlyeq$) on 2^N in the following way,

$$\forall S, T \in 2^N, S \succcurlyeq T \Rightarrow T - \succcurlyeq S.$$

Relation $-\succcurlyeq$ is defined on elements of 2^N and is the reversal of $\succcurlyeq$, so, it is an element of $\mathcal{T}(2^N)$.

We now introduce some elementary properties for transformations defined according to the previous definitions. Let N and X be two sets of individuals, $\succcurlyeq$ an element of $\mathcal{T}(2^N)$ such that $\text{vot}(\succcurlyeq) = \{S_1, ..., S_l\}$. We have that:

- If $X \cap N = \emptyset$, then $\text{vot}(\succcurlyeq^X) = \{S_1 \cup X, ..., S_l \cup X\}$ and $\geq^{S_k \cup X} = \geq^{S_k}$ for every k in $[\![1, l]\!]$;
- $\text{vot}(-\succcurlyeq) = \{S_1, ..., S_l\}$ and if we denote $\geq^{S_k}{}'$ the preferences of the voter S_k in consideration of $-\succcurlyeq$, then if i and j are in N, $i \geq^{S_k}{}' j$ if and only if $j \geq^{S_k} i$ for every k in $[\![1, l]\!]$;
- If $X \cap N = \emptyset$ and R is a solution to the social ranking problem that satisfies Separability$_{SR}$ and Cancellation$_{SR}$, then for every individuals k and l in N $kR(\succcurlyeq^X)l$ if and only if $kR(\succcurlyeq)l$;
- If R is a solution to the social ranking problem that satisfies Separability$_{SR}$ and Cancellation property$_{SR}$, then for every individuals k and l in N if $kP(\succcurlyeq)l$ then $lP(-\succcurlyeq)k$;

Let's now define notions of *amplifications of coalitional preorders.*

Definition 3.17. Let N be a set of individuals and $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$, with $\text{vot}(\succcurlyeq) = \{S^1, ..., S^l\}$ its voters, $\text{can}(\succcurlyeq) = \{i_1, ..., i_m\}$ its candidates and $\mathcal{P}_{\succcurlyeq} = \{\geq^{S_1}, ..., \geq^{S_l}\}$ its preference profile as defined in Section 3. Let i be a candidate for $\succcurlyeq$. We call **amplification of** $\succcurlyeq$ **in relation to** i and denote $\succcurlyeq^{\hat{i}}$ a coalitional preorder in $\mathcal{T}(2^{\hat{N}})$ of the form $\succcurlyeq^{\hat{i}} = \succcurlyeq^1 \boxplus \succcurlyeq^2 \boxplus ... \boxplus \succcurlyeq^{(m-1)!}$, where:

- m is the cardinal of $\text{can}(\succcurlyeq)$;
- $\hat{N} = \{1, ..., n(m-1)!\}$ is a set of individuals;
- For k in range $[\![1, (m-1)! - 1]\!]$ we denote T^k the subset of $\hat{N}$ $\{kn+1, ..., (k+1)n\}$ and $\succcurlyeq^k = \sigma_k(\succcurlyeq^{T_k})$, (with $\succcurlyeq^{T_k}$ defined as in 3.15 and σ_k a permutation on N such that $\sigma_k(i) = i$).
- by convention $\sigma_{(m-1)!}(i)$ is the identity permutation; so, $\succcurlyeq$ coincides with $\succcurlyeq^{(m-1)!}$.

Given an amplification $\succcurlyeq^{\hat{i}}$ the set of voters $V_{\hat{i}} = \text{vot}(\succcurlyeq^{\hat{i}})$ is $V_{\hat{i}} = \{S^1, ..., S^l, S^1 \cup T^1, ..., S^l \cup T^1, ..., S^1 \cup T^{(m-1)!-1}, ..., S^l \cup T^{(m-1)!-1}\}$.

We illustrate an instance of amplification of a coalitional preorder in the following example.

Example 3.18. Let us consider the set of individuals $N = \{1, 2, 3\}$ and the coalitional preorder $\succcurlyeq$ such that

$$\{1, 2\} \succ \{1, 3\}, \{1\} \succ \{2\}.$$

Borda's scores for $\succcurlyeq$ are $\beta(1) = 1$, $\beta(2) = 0$ and $\beta(3) = -1$.

We want to define an amplification of it in relation to 1. The voters of $\succcurlyeq$ are $\emptyset$ and $\{1\}$. We place ourselves on $\hat{N} = \{1, 2, 3, 4, 5, 6\}$ in order to define new voters $\{4, 5, 6\} = \emptyset \cup \{4, 5, 6\}$ and $\{1, 4, 5, 6\} = \{1\} \cup \{4, 5, 6\}$. The candidates of $\succcurlyeq$ are 1, 2 and 3 so the only two permutations on them that fix 1 (i.e., such that $\sigma_k(1) = 1, k = 1, 2$) are the transposition (23) and the identity.

Then an amplification of $\succcurlyeq$ in relation to 1 is the coalitional preorder $\succcurlyeq'$ on $\{1, 2, 3, 4, 5, 6\}$ such that

$$\{1, 3, 4, 5, 6\} \succ^{\hat{i}} \{1, 2, 4, 5, 6\}, \{1, 4, 5, 6\} \succ^{\hat{i}} \{3, 4, 5, 6\}, \{1, 2\} \succ^{\hat{i}} \{1, 3\},$$

$$\{1\} \succ^{\hat{i}} \{2\}.$$

Where, with notations of Definition 3.17, $T_1 = \{4, 5, 6\}$, $\sigma_1 = (23)$, $\sigma_2 = Id$, and where $\{1, 3, 4, 5, 6\} \succ^{\hat{i}} \{1, 2, 4, 5, 6\}, \{1, 4, 5, 6\} \succ^{\hat{i}} \{3, 4, 5, 6\}$ corresponds to $\sigma_1(\succcurlyeq^{T_1})$ and $\{1, 2\} \succ^{\hat{i}} \{1, 3\}, \{1\} \succ^{\hat{i}} \{2\}$ to $\sigma_2(\succcurlyeq)$.

Borda's scores for $\succcurlyeq^{\hat{1}}$ are $\beta^{\hat{1}}(1) = 2$, $\beta^{\hat{1}}(2) = -1$, $\beta^{\hat{1}}(3) = -1$ and $\beta^{\hat{1}}(k) = 0$ for k in $\{4, 5, 6\}$.

Proposition 3.19. *Let N be a set of individuals, $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$, $m = can(\succcurlyeq)$ be the cardinal of its candidates set and i be in $can(\succcurlyeq)$. Then an amplification $\succcurlyeq^{\hat{i}}$ of $\succcurlyeq$ in relation to i has the following properties:*

(1) $\beta^{\succcurlyeq^{\hat{i}}}(i) = (m-1)!\, \beta^{\succcurlyeq}(i)$.
(2) $\beta^{\succcurlyeq^{\hat{i}}}(j) = -(m-2)!\, \beta^{\succcurlyeq}(i)$ *for all j in $can(\succcurlyeq) \backslash \{i\}$.*
This relation (2) also holds if i isn't a candidate and we consider the set of all permutations on candidates and define $\succcurlyeq^{\hat{i}}$ in the same way as for a candidate.

Proof. See Appendix. □

Example 3.20. Let us consider another example to understand the amplification in relation to an individual that is not a candidate.

Let us consider the set of individuals $N = \{1, 2, 3\}$ and the coalitional preorder $\succcurlyeq$ such that

$$\{1, 2\} \succ \{2, 3\}, \{1, 2, 3\} \succ \emptyset.$$

Borda's scores for $\succcurlyeq$ are $\beta(1) = 1$, $\beta(2) = 0$ and $\beta(3) = -1$.

We want to define its amplification in relation to 2. The unique voter of $\succcurlyeq$ is $\{2\}$. We place ourselves on $\hat{N} = \{1, 2, 3, 4, 5, 6\}$ in order to define new voter $\{2, 4, 5, 6\} = \{2\} \cup \{4, 5, 6\}$. The candidates of $\succcurlyeq$ are 1 and 3 so we consider the set of all permutations on $\{1, 3\}$, $\{(13), Id\}$.

Then the coalitional preorder $\succcurlyeq^{\hat{1}}$ on $\{1, 2, 3, 4, 5, 6\}$ such that $\{2, 3, 4, 5, 6\} \succ^{\hat{2}} \{1, 2, 4, 5, 6\}$, $\{1, 2, 3, 4, 5, 6\} \succ^{\hat{2}} \emptyset$, $\{1, 2\} \succ^{\hat{2}} \{2, 3\}, \{1, 2, 3\} \succ^{\hat{2}} \emptyset$ is an amplification of $\succcurlyeq$ in relation to 2.

According to notations of Definition 3.17, we have $T_1 = \{4, 5, 6\}$, $\sigma_1 = (23)$, $\sigma_2 = Id$, $\{2, 3, 4, 5, 6\} \succ^{\hat{2}} \{1, 2, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\} \succ^{\hat{2}} \{4, 5, 6\}$, is obtained using $\sigma_1(\succcurlyeq^{T_1})$ and $\{1, 2\} \succ^{\hat{2}} \{2, 3\}, \{1, 2, 3\} \succ^{\hat{2}} \emptyset$ is obtained using $\sigma_2(\succcurlyeq)$.

Borda's scores for $\succcurlyeq^{\hat{1}}$ are $\beta^{\hat{2}}(1) = \beta^{\hat{2}}(2)$ and $\beta^{\hat{2}}(3) = 0$.

Proposition 3.21. *Let R be a solution for the social ranking that verifies Neutrality$_{SR}$ and Separability$_{SR}$. Let N be a set of individuals, $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$, m be the cardinal of its candidates set and i be in $\succcurlyeq$ candidates. $\hat{N}$ is defined as in Definition 3.17.*

If for every j in N, $iR^N(\succcurlyeq)j$. Then for every j in N, $iR^N(\succcurlyeq^{\hat{i}})j$. Furthermore, if there is a k in N such that $iP^N(\succcurlyeq)k$ (and $iR^{\hat{N}}(\succcurlyeq)j$ for every j in $\hat{N}$) then $iP^{\hat{N}}(\succcurlyeq^{\hat{i}})j$ for every j in $\hat{N}$.

Proof. This is a direct consequence of the Neutrality$_{SR}$ and Separability$_{SR}$ axioms. □

Sometimes we want to maintain the "intensity" of the social relation between two specific candidates, while homogenizing the social role of the others. For this purpose we define amplification of a preorder in relation to two individuals.

Definition 3.22. Let N be a set of individuals and $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$. We define $\succcurlyeq^{\hat{ij}}$ an amplification of $\succcurlyeq$ in relation to individuals i and j in the same way as we define an amplification in relation to one individual in Definition 3.17. The only difference is that we only consider permutations that fix both i and j and a set of individuals $\hat{\hat{N}} = \{1, ..., n(m-2)!\}$.

Proposition 3.23. *Let N be a set of individuals, $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$, m be the cardinal of its candidates set can($\succcurlyeq$) and i be in can($\succcurlyeq$). Then an amplification $\succcurlyeq^{\hat{ij}}$ of $\succcurlyeq$ in relation to i and j has the following properties:*

(1) *$\beta^{\succcurlyeq^{\hat{ij}}}(i) = (m-2)!\,\beta^{\succcurlyeq}(i)$ and $\beta^{\succcurlyeq^{\hat{ij}}}(j) = (m-2)!\,\beta^{\succcurlyeq}(j)$;*
(2) *$\beta^{\succcurlyeq^{\hat{ij}}}(k) = -(m-3)!\,(\beta^{\succcurlyeq}(i)+\beta^{\succcurlyeq}(j))$ for all k in can($\succcurlyeq$)$\backslash\{i,j\}$.*
And if R is a solution for the social ranking problem verifying Neutrality$_{SR}$ and Separability$_{SR}$ then it holds that
(3) *$iR^N(\succcurlyeq)j \Rightarrow iR^{\hat{\hat{N}}}(\succcurlyeq^{\hat{i,j}})j$, $iP^N(\succcurlyeq)j \Rightarrow iP^{\hat{\hat{N}}}(\succcurlyeq^{\hat{i,j}})j$, $jP^N(\succcurlyeq)i \Rightarrow jP^{\hat{\hat{N}}}(\succcurlyeq^{\hat{i,j}})j$ and $jR(\succcurlyeq)i \Rightarrow jR^{\hat{\hat{N}}}(\succcurlyeq^{\hat{i,j}})i$*

Proof. See appendix. □

Example 3.24. Let's consider a variation of Example 3.18 to understand an amplification in relation to two individuals (here two candidates but the idea is the same with one candidate and one non candidate individual and with two non candidates individuals).

Let us consider the set of individuals $N = \{1, 2, 3, 4\}$ and the coalitional preorder $\succcurlyeq$ such that

$$\{1,2\} \succ \{1,3\} \succ \{1,4\}, \{1\} \succ \{2\}, \{3\} \succ \{4\}.$$

Borda's scores are $\beta(1) = 1$, $\beta(2) = 1$, $\beta(3) = 1$ and $\beta(4) = -3$.

We want to define an amplification of $\succcurlyeq$ in relation to 2 and 4. The voters of $\succcurlyeq$ are $\emptyset$ and $\{1\}$. We place ourselves on $\hat{N} = \{1, 2, 3, 4, 5, 6, 7\}$ in order to define new voters $\{5, 6, 7\} = \emptyset \cup \{5, 6, 7\}$ and $\{1, 5, 6, 7\} = \{1\} \cup \{5, 6, 7\}$. The candidates of $\succcurlyeq$ are 1, 2, 3 and 4 so the only two permutations on them that fix both 2 and 4 are the transposition (13) and the identity. Then, the coalitional preorder $\succcurlyeq^{\hat{2,4}}$ on $\{1, 2, 3, 4, 5, 6\}$ such that

$$\{2,3,5,6,7\} \succ^{\hat{2,4}} \{1,3,5,6,7\} \succ^{\hat{2,4}} \{3,4,5,6,7\},$$

$$\{3,5,6,7\} \succ^{\hat{2,4}} \{2,5,6,7\}, \{1,5,6,7\} \succ^{\hat{2,4}} \{4,5,6,7\},$$

$$\{1,2\} \succ^{\hat{2,4}} \{1,3\} \succ^{\hat{2,4}} \{1,4\}, \{1\} \succ^{\hat{2,4}} \{2\}, \{3\} \succ^{\hat{2,4}} \{4\},$$

is an amplification $\succcurlyeq$ in relation to 2, 4.

Borda's scores are $\beta^{\hat{2,4}}(1) = 2$, $\beta^{\hat{2,4}}(2) = 2$, $\beta^{\hat{2,4}}(3) = 2$, and $\beta^{\hat{2,4}}(4) = -6$

4 MAIN RESULT

In this section, we prove that CP-Borda solution to the social ranking problem is the only solution that verifies Neutrality$_{SR}$, Separability$_{SR}$, Desirability$_{SR}$ and Cancellation$_{SR}$. Our proof is divided in three lemmas. For the first two lemmas, we use the proof scheme introduced by B. Hansson and H. Sahlquist in [10], and their heavy use of the notion of amplification of a profile, here transposed to amplification of a coalitional order. For the last lemma, we use the idea of S. Nitzan and A. Rubinstein in their article [16] to, once we proved that a solution verifying our set of axioms only depends on Borda's scores, construct a more usable profile to prove that the final ranking is in the "right order". Their article also use Monotonicity instead of Faithfulness used by H.P. Young [21] and B. Hansson and H. Sahlquist [10], which is closer to Desirability$_{SR}$.

Our first goal is to show in Lemma 4.2 that if a solution R to the social ranking problem satisfies Neutrality$_{SR}$, Separability$_{SR}$, and Cancellation$_{SR}$, then for every set of individuals N and preorders $\succcurlyeq_1$ and $\succcurlyeq_2$ in $\mathcal{T}(2^N)$ such that for every i in N, $\beta^{\succcurlyeq_1}(i) = \beta^{\succcurlyeq_2}(i)$, then $R(\succcurlyeq_1) = R(\succcurlyeq_2)$. The first step to do so is to prove that if a coalitional preorder $\succcurlyeq$ is such that $\beta^{\succcurlyeq}(i) = 0$ for every individual i then $R(\succcurlyeq) = \sim$. This is proved in Lemma 4.1, where we amplify a coalitional order $\succcurlyeq$ verifying the nullity of Borda's score in relation to the individual that is supposed to be the "best ranked one" in the image of $\succcurlyeq$ by a solution R verifying our set of axioms, and then showing that such a best ranked individual is not strictly better ranked than any other individual. In the following, for any $\succcurlyeq_2$ in $\mathcal{T}(2^N)$, let $\mathfrak{S}_{\hat{i}} = \{\sigma \in \mathfrak{S}(\text{can}(\succcurlyeq)), \sigma(i) = i\} = \{\sigma_1, ..., \sigma_{(m-1)!}\}$, where $\mathfrak{S}(\text{can}(\succcurlyeq))$ is the set of all permutations on can($\succcurlyeq$) and $\sigma_{(m-1)!}$ is the identity permutation.

Lemma 4.1. *Let N be a set of individuals and $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$ such that $\beta(i) = 0$ for every i in N. If R is a solution for the social ranking problem that verifies axioms Neutrality$_{SR}$, Separability$_{SR}$ and Cancellation$_{SR}$, then $R(\succcurlyeq) = \sim_N$.*

Proof. Let N be a set of individuals, $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$ such that $\beta(i) = 0$ for every i in N, and R be a solution for the social ranking problem that verifies axioms Neutrality$_{SR}$, Separability$_{SR}$, and Cancellation$_{SR}$. Let m be the cardinal of the set of candidates for $\succcurlyeq$ and $\hat{N} = \{1, ..., n(m-1)!\}$.

Let's suppose that $R(\succcurlyeq) \neq \sim$. $R(\succcurlyeq)$ is a total transitive order so there is an individual i in $\hat{N}$ such that $iR(\succcurlyeq)j$ for every j in $\hat{N}$. Since we supposed that $R(\succcurlyeq) \neq \sim_N$, there is an element k in N such that $iP(\succcurlyeq)k$. We consider an amplification $\succcurlyeq^{\hat{i}}$. By Proposition 3.21, $iP(\succcurlyeq^{\hat{i}})k$.

Let a, b be elements of $\hat{N}$. The following facts hold:

- If a or b are not in can($\succcurlyeq$) then $\pi_{ab}(\succcurlyeq^{\hat{i}}) = \pi_{ba}(\succcurlyeq^{\hat{i}}) = 0$;
- if a and b are in can($\succcurlyeq$)$\backslash\{i\}$, then

$$\pi_{ab}(\succcurlyeq^{\hat{i}}) = \sum_{\sigma \in \mathfrak{S}_{\hat{i}}} \pi_{\sigma^{-1}(a)\sigma^{-1}(b)}(\succcurlyeq)$$

- if σ is in $\mathfrak{S}_{\hat{i}}$ then σ^{-1} also is and by symmetry of $\mathfrak{S}_{\hat{i}}$ regarding individuals different from i, $\mathfrak{S}_{\hat{i}} = \{\sigma \circ \tau_{ab}, \sigma \in \mathfrak{S}_{\hat{i}}\}$, where τ_{ab} is the permutation that exchanges a and b. So, from the formula at the previous point, it follows then that $\pi_{ab}(\succcurlyeq^{\hat{i}}) = \pi_{ba}(\succcurlyeq^{\hat{i}})$.

- Let a be an element of $\mathrm{can}(\succcurlyeq)$. If $\mathrm{vot}(\succcurlyeq) = \{S_1, ..., S_l\}$, we consider the two sets $i^- = \{(k,u), k >^{S_u} i\}$ and $i^+ = \{(k,u), i >^{S_u} k\}$.

$$\begin{aligned}\pi_{ia}(\succcurlyeq^{\hat{i}}) &= \sum_{u=1}^{l}\sum_{k\in N} |\{\sigma \in \mathfrak{S}_i, \sigma(b) = k, (k,u) \in i^-\}| \\ &= \sum_{u=1}^{l}\sum_{k\in N} |\{\sigma \in \mathfrak{S}_i, \sigma(b) = k\}| \delta_{(k,l)\in i^-} \\ &= \sum_{u=1}^{l}\sum_{k\in N} (m-1)!\ \delta_{(k,l)\in i^-} \\ &= |i^-| \times (m-1)!.\end{aligned}$$

  Where $\delta_{(k,l)\in i^-}$ is the Kronecker symbol that is equal to 1 if $(k,l) \in i^-$ and 0 else.
  We can show in a similar way that $\pi_{ai}(\succcurlyeq^{\hat{i}}) = |i^+| \times (m-1)!$. However, since $\beta^{\succcurlyeq}(i) = 0$, we have $|i^+| = \sum_{j\in N} \pi_{ij}(\succcurlyeq) = \sum_{j\in N} \pi_{ji}(\succcurlyeq) = |i^-|$. And then $\pi_{ai}(\succcurlyeq^{\hat{i}}) = \pi_{ia}(\succcurlyeq^{\hat{i}})$.

We showed that for all a, b in $\hat{N}$, we have $\pi_{ab}(\succcurlyeq^{\hat{i}}) = \pi_{ba}(\succcurlyeq^{\hat{i}})$. Since R verifies Cancellation$_{SR}$, $R(\succcurlyeq^{\hat{i}}) =\sim$. This is in contradiction with $iP(\succcurlyeq^{\hat{i}})k$ and therefore, $R(\succcurlyeq) =\sim$. □

Using Lemma 4.1 and the fact that R verifies Separability$_{SR}$, we can prove that R depends only on Borda's scores of individuals.

Lemma 4.2. *Let N be a set of individuals and $\succcurlyeq_1$ and $\succcurlyeq_2$ be elements of $\mathcal{T}(2^N)$ such that $\beta^{\succcurlyeq_1}(i) = \beta^{\succcurlyeq_2}(i)$ for every i in N, then if R is a solution for the social ranking problem that verifies axioms Neutrality$_{SR}$, Separability$_{SR}$ and Cancellation$_{SR}$, then $R(\succcurlyeq_1) = R(\succcurlyeq_2)$.*

Proof. Let's suppose that there exist i and j in N such that $iR(\succcurlyeq_1)j$ and $jP(\succcurlyeq_2)i$ We consider $\succcurlyeq_1$ and $\succcurlyeq_2$ on the set $N' = \{1, ..., 2n\}$ and the subset $X = \{n+1, ..., 2n\}$. Let $\succcurlyeq = \succcurlyeq_1 \boxplus (-\succcurlyeq_2^X)$. By Separability$_{SR}$ we have $iP(\succcurlyeq)j$. For every i in $N' \backslash N$, $\beta^{\succcurlyeq}(i) = 0$, for every i in N,

$$\beta^{\succcurlyeq}(i) = \beta^{\succcurlyeq_1}(i) + \beta^{-\succcurlyeq_2^X}(i) = \beta^{\succcurlyeq_1}(i) - \beta^{\succcurlyeq_2}(i) = 0.$$

By Lemma 4.1, $R(\succcurlyeq) = \sim$, which contradicts $iP(\succcurlyeq)j$. Therefore, $R(\succcurlyeq_1) = R(\succcurlyeq_2)$ and R is only defined by Borda's scores. □

The final step is now to show that R depends on Borda's score in the "right way" thanks to Desirability$_{SR}$, meaning that it orders individuals by increasing Borda's score. To do this we use an amplification of a coalitional order $\succcurlyeq$ in relation to two individuals, to focus on the relative ranking between these two specific individuals in the image of $\succcurlyeq$ by R while smoothing out other individuals interventions.

Lemma 4.3. *Let N be a set of individuals, $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$ and i and j be individuals such that $\beta(i) \geq \beta(j)$ (resp. $\beta(i) > \beta(j)$), then if R is a solution for the social ranking problem that verifies axioms Neutrality$_{SR}$, Separability$_{SR}$, Desirability$_{SR}$ and Cancellation property$_{SR}$, $iR(\succcurlyeq)j$ (resp. $iP(\succcurlyeq)j$).*

Proof. Let R be a solution to the social ranking problem that verifies Neutrality$_{SR}$, Separability$_{SR}$, Desirability$_{SR}$ and Cancellation$_{SR}$, $N = \{1, ..., n\}$ be a set of individuals, $\succcurlyeq$ be an element of $\mathcal{T}(2^N)$ and i and j be individuals such that $\beta(i) \geq \beta(j)$ (resp. $\beta(i) > \beta(j)$). We treat the case where both i and j are candidates for $\succcurlyeq$ but other cases can be treated in the same way. For easier notations, we will consider that $i = 1$ and $j = 2$ and that $\mathrm{can}(\succcurlyeq) = \{1, ..., m\}$, where m is the cardinal of the set of candidates for $\succcurlyeq$. Let's denote p the value $\beta(1) - \beta(2)$, and (q, r) the euclidean division of $|\frac{\beta(1)+\beta(2)}{2}|$ by $(m-2)$ if p is even and the euclidean division of $|\frac{\beta(1)+\beta(2)-1}{2}|$ by $(m-2)$ if p is odd. We consider $\succcurlyeq$ in $\tilde{N} = \{1, ..., ((p/2) + q + 2)n\}$ if p is even and in $\tilde{N} = \{1, ..., (((p+1)/2) + q + 3)n\}$ if p is odd.

Due to the lack of space we only present the case where p is even but the odd case can be trated in a similar way.

Our objective is to build a coalitional preorder $\succcurlyeq^{\Delta}$ on which we can easily use the Desirability$_{SR}$ axiom to obtain a preference between 1 and 2. We want 1 and 2 to have the same Borda's scores in $\succcurlyeq^{\Delta}$ and in $\succcurlyeq$ to be able to use Lemma 4.2 on amplified profiles (a numerical example of this step is presented in the Appendix).

We consider the coalitional preorder $\succcurlyeq^{\Delta}$ in $\mathcal{T}(2^{\tilde{N}})$ defined in the following way: for k in range $[[1, (p/2) + q + 1]]$ we define $T^k = \{kn + 1, ..., (k+1)n\}$.

For k in range $[[1, p/2]]$ we define the coalitional preorders $\succcurlyeq_k$ such that

$$T^k \cup \{1\} \succ^k T^k \cup \{2\}.$$

These $p/2$ first relations will be used to ensure the "advance" from 1 on 2 in the preorder we're building.

Then, if $\beta(1) + \beta(2)$ is positive, we define for k in range $[[(p/2) + 1, (p/2) + q]]$ the coalitional preorders $\succcurlyeq_k$ such that

$$T^k \cup \{1\} \sim_k T^k \cup \{2\} \succ_k T^k \cup \{3\} \succ_k ... \succ_k T^k \cup \{m\}.$$

If $\beta(1) + \beta(2)$ is negative, we consider the opposite relations. We define $\succcurlyeq_{(p/2)+q+1}$ as

$$T^k \cup \{1\} \sim_{(p/2)+q+1} T^k \cup \{2\} \succ_{(p/2)+q+1}$$
$$T^{(p/2)+q+1} \cup \{3\} \sim_{(p/2)+q+1} ... \sim_{(p/2)+q+1} T^{(p/2)+q+1} \cup \{r+2\}.$$

These relations will be used to ensure that Borda scores of 1 and 2 are the same as in $\succcurlyeq$. Indeed, having the right gap isn't sufficient to use Lemma 4.3, scores have to be the same.

In the case where $q = 0$, we define the relation $T^{(p/2)+1} \cup \{3\} \sim_{(p/2)+1} ... \sim_{(p/2)+1} T^{(p/2)+1} \cup \{m\}$. This ensure that every candidates of $\succcurlyeq$ are candidates in $\succcurlyeq^{\Delta}$ and that we can correctly use amplification.

We now define $\succcurlyeq^{\Delta}$ as the preorder $\succcurlyeq_1 \boxplus ... \boxplus \succcurlyeq_{(p/2)+q+1}$. By construction, $\beta^{\succcurlyeq^{\Delta}}(1) = \beta^{\succcurlyeq}(1)$, $\beta^{\succcurlyeq^{\Delta}}(2) = \beta^{\succcurlyeq}(2)$ and $\mathrm{can}(\succcurlyeq^{\Delta}) = \mathrm{can}(\succcurlyeq)$. Then by Proposition 3.23 we have $\beta^{\succcurlyeq^{\widehat{1,2}}}(k) = \beta^{\succcurlyeq^{\Delta \widehat{1,2}}}(k)$ for every k in $\hat{\hat{N}} = \{1, ..., n(m-2)!\}$, and by Lemma 4.3 we have $R(\succcurlyeq^{\widehat{1,2}}) = R(\succcurlyeq^{\Delta \widehat{1,2}})$.

If $p > 0$, we have 1 strictly more desirable than 2 in $\succcurlyeq^{\Delta}$ and then in $\succcurlyeq^{\Delta \widehat{1,2}}$ by Proposition 3.23. Since R verifies Desirability$_{SR}$, $1P(\succcurlyeq^{\Delta \widehat{1,2}})2$ and so $1P(\succcurlyeq^{\widehat{1,2}})2$. According to Proposition 3.23 and the fact that $R(\succcurlyeq)$ is total, it follows that $1P(\succcurlyeq)2$. If p is equal to 0, then we have $1I(\succcurlyeq)2$ in a similar way. This concludes the proof. □

Theorem 4.4. *The only solution for the social ranking problem that verifies axioms Neutrality$_{SR}$, Separability$_{SR}$, Desirability$_{SR}$ and Cancellation property$_{SR}$ is CP-Borda solution.*

Proof. It directly follows from Lemmas 3.14 and 4.3. □

4.1 Axioms independence

We now show that the four axioms are logically independent.

• Given a set of individuals $N = \{1, ..., n\}$ and an element $\succcurlyeq$ of $\mathcal{T}(2^N)$, we define the following score β', for every individual i:

$$\beta'(i) = \sum_{j \in N} j(\pi_{ij}(\succcurlyeq) - \pi_{ji}(\succcurlyeq)).$$

The solution for the social ranking problem R such that given a set of individuals N and an element $\succcurlyeq$ of $\mathcal{T}(2^N)$, $iR(\succcurlyeq)j$ if $\beta'(i) \geq \beta'(j)$, verifies Separability$_{SR}$, Desirability$_{SR}$ and Cancellation$_{SR}$ but does not satisfy Neutrality$_{SR}$.

• Given a set of individuals $N = \{1, ..., n\}$ and an element $\succcurlyeq$ of $\mathcal{T}(2^N)$, we define the following score β', for every individual k, :

$$\beta'(k) = \begin{cases} 0 & \text{if } \pi_{ij}(\succcurlyeq) = \pi_{ji}(\succcurlyeq)\ \forall\, i, j \in N \text{ or if } \emptyset \notin \text{ vot}(\succcurlyeq) \\ \beta^{\succcurlyeq}(k) & \text{if } \exists i \in N, i \text{ is more desirable than } k \\ & \text{ or } k \text{ is more desirable than } i \\ \beta_{\emptyset}(k) & \text{else} \end{cases}$$

where for $S \in \text{vot}(\succcurlyeq)$, $\beta_S(k)$ is the borda score of k in the preference order of voter S. The solution for the social ranking problem R defined such that given a set of individuals N and an element $\succcurlyeq$ of $\mathcal{T}(2^N)$, $iR(\succcurlyeq)j$ if $\beta'(i) \geq \beta'(j)$, verifies Neutrality$_{SR}$, Desirability$_{SR}$ and Cancellation$_{SR}$ but does not satisfy Separability$_{SR}$.

• Given a set of individuals $N = \{1, ..., n\}$ and an element $\succcurlyeq$ of $\mathcal{T}(2^N)$, the solution for the social ranking problem R defined such that $iR(\succcurlyeq)j$ if $\beta^{\succcurlyeq}(j) \geq \beta^{\succcurlyeq}(i)$ verifies Neutrality$_{SR}$, Separability$_{SR}$ and Cancellation$_{SR}$ but does not satisfy Desirability$_{SR}$.

• Given a set of individuals $N = \{1, ..., n\}$ and an element $\succcurlyeq$ of $\mathcal{T}(2^N)$, we define the following score β', for every individual i:

$$\beta'(i) = \sum_{j \in N} \pi_{ij}(\succcurlyeq).$$

The solution for the social ranking problem R defined such that given a set of individuals N and an element $\succcurlyeq$ of $\mathcal{T}(2^N)$, $iR(\succcurlyeq)j$ if $\beta'(i) \geq \beta'(j)$, verifies Neutrality$_{SR}$, Separability$_{SR}$ and Desirability$_{SR}$ but does not satisfy Cancellation$_{SR}$.

5 CP-BORDA AND CP-MAJORITY

Since CP-majority solution to the social ranking problem can be seen as a transposition of Condorcet's voting rule to the social ranking problem, we state some results that echo those known on Condorcet and Borda's voting rules.

Definition 5.1. Given a set of individuals $N = \{1, ..., n\}$, the **CP-majority solution to the social ranking problem on** N, R^N_{CP} is the social ranking solution on N such that for every coalitional preorder $\succcurlyeq$ in $\mathcal{T}(2^N)$ and every individuals i and j in N, $iR^N_{CP}j$ if $\pi_{ij}(\succcurlyeq) \geq \pi_{ji}(\succcurlyeq)$. $R_{CP} = \{R^N_{CP}, N = \{1, ..., n\}, n \in \mathbb{N}\}$ is the **CP-majority** solution to the social ranking problem.

It's important to first remind that CP-majority is defined on a domain different from the CP-Borda's one. CP-majority takes in input a complete preorder on coalitions and produces a complete, reflexive but not necessarily transitive binary relation. Indeed, CP-majority verifies Neutrality$_{SR}$, Desirability$_{SR}$ (used in its axiomatic characterization in [1]) and it is easy to check that it also satisfies Cancellation$_{SR}$. Actually, if we loosen up a bit the frame of definition, we can show that this solution also verifies Separability$_{SR}$. The fact that CP-Majority does not always produce a transitive binary relation on the individuals ensures that there is no contradiction with Theorem 4.4. To be able to make meaningful comparisons, next we will consider complete preorders on coalitions on which CP-majority does in fact produce a complete preorder.

• CP-Borda does not necessary rank a CP-majority winner first.

In the same way that Borda's voting rule can fail to elect a Condorcet winner, even if a CP-majority winner exists it is not assured to be ranked first by CP-Borda. This mainly comes from the sensibility to discrepancy between individuals in voters preferences. Even if more voters prefer an individual i to another one j, if voters that prefer j express a stronger appreciation also in relation to other individuals, j can be ranked above i in the final preorder. The following preorder illustrate this property: $\{2\} \succ \{3\} \succ \{4\} \succ \{5\} \succ \{1\} \succ \{1,2\} \succ \{1,3\} \succ \{2,3\} \succ \{2,4\} \succ \{1,4\} \succ \{3,4\} \succ \{2,5\} \succ \{1,5\} \succ \{3,5\} \succ \{4,5\} \succ \{1,2,3\} \succ \{1,2,4\} \succ \{1,2,5\} \succ \{1,3,4\} \succ \{1,3,5\} \succ \{1,4,5\} \succ \{2,3,4\} \succ \{2,3,5\} \succ \{2,4,5\} \succ \{3,4,5\} \succ \{1,2,3,4\} \succ \{1,2,3,5\} \succ \{1,2,4,5\} \succ \{1,3,4,5\} \succ \{2,3,4,5\} \succ \{1,2,3,4,5\}$. We have $\pi_{12} = 5$, $\pi_{13} = \pi_{14} = \pi_{15} = 7$, $\pi_{21} = 3$, $\pi_{23} = \pi_{24} = \pi_{25} = \pi_{34} = \pi_{35} = \pi_{45} = 8$, $\pi_{31} = \pi_{41} = \pi_{51} = 1$ and $\pi_{32} = \pi_{42} = \pi_{52} = \pi_{43} = \pi_{53} = \pi_{54} = 0$.

According to this, the CP-majority solution outputs the following preorder on $\{1,2,3,4,5\}$: $1 > 2 > 3 > 4 > 5$.

However, Borda's scores are: $\beta(1) = 20$, $\beta(2) = 22$, $\beta(3) = 2$, $\beta(4) = -14$, $\beta(5) = -30$, and so CP-Borda outputs the following preorder on $\{1,2,3,4,5\}$: $2 > 1 > 3 > 4 > 5$.

• CP-Borda cannot rank first a CP-majority loser.

Let N be a set of individual and $\succcurlyeq$ be in $\mathcal{T}(2^N)$. Let's suppose that i is a CP-loser (i.e., for every j in $N \backslash \{i\}$, $\pi_{ij}(\succcurlyeq) < \pi_{ji}(\succcurlyeq)$), then

$$\beta(i) = \sum_{j \in N} \pi_{ij}(\succcurlyeq) - \pi_{ji}(\succcurlyeq) < 0.$$

Since the sum of all Borda's score is equal to 0, then there is at least one individual that has a strictly positive Borda score and so j can't be ranked in first place.

6 CONCLUSION

In this paper, we introduce and axiomatically characterize a novel solution for the social ranking problem that mimics the well known Borda's voting rule in a coalitional setting. The definition of this solution is based on the notion of *Ceteris Paribus* comparison for coalitions, that was already presented in related literature [11]. This notion has been used to provide a set of meaningful axioms characterizing our CP-Borda solution that can also be employed to make a comparisons with CP-majority, another solution from the literature [11].

Although the convincing interpretation of the axioms used in this paper suggests that our definition of the CP-Borda solution is sound, we note that alternative ways to define a coalitional Borda score are possible, perhaps even using more general information provided by a coalitional preorder, such as the number of dominated coalitions (which are not necessarily those involved in CP-comparisons).

Another interesting research direction would be to deepen the analytical comparison with the CP-majority, examining quantitative indicators such as the proportion of coalitional preorders in which CP-Borda fails to rank a CP-majority winner first.

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